Worksheet on Chapters 1 and 2

1 Functions

Problem 1. Consider the function $f(x) = 5x + 7$. Which of the following is true

1. The graph of $f(x)$ does not intersect the line the $y = x$ and therefore it does not have an inverse function

2. The inverse function $f^{-1}(x)$ exists and therefore its graph never touches the graph of $f(x)$

3. The graph of $f^{-1}(x)$, the graph of $f(x)$ and the line $y = x$ all meet in exactly one point

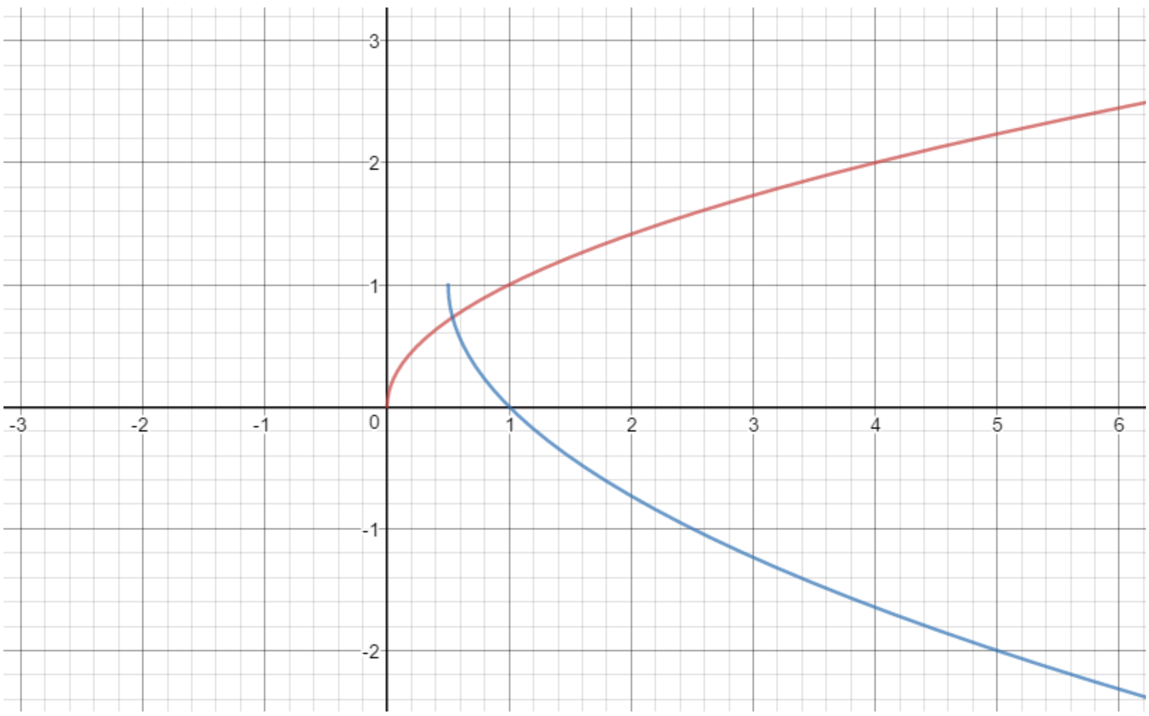
4. The function $f^{-1}(x)$ exists, but only after restricting the domain of $f(x)$ appropriately

Solution: the function is invertible on the whole domain ($\mathbb{R}$) and $f^{-1}(x) = \frac{1}{5}x - \frac{7}{5}$. So the answers 2. and 4. are wrong. The graph of $f(x)$ does intersect the line $y = x$, so the answer 1. is wrong. In any case, having inverse has nothing to do with the graph intersecting the line $y = x$.
So the answer is 3., which is also easily verifiable directly.
Problem 2. Find the implied domains of the following functions

1. $f(x) = \sqrt{73 - x} - \sqrt{37 + x}$

2. $f(x) = 5\ln(x - 6)$

Solution: 1. The domain for this function is the intersection of the domains of $\sqrt{73 - x}$ and $\sqrt{37 + x}$. The domain of $\sqrt{73 - x}$ is $(-\infty, 73]$ and the domain of $\sqrt{37 + x}$ is $[-37, +\infty)$. Therefore, the answer is $[-37, 73]$.
2. The function is defined whenever the argument of ln is positive, i.e. for $x \in (6, +\infty)$.
Problem 3. Sketch the graph of $y = -\sqrt{2x - 1} + 1$ starting from the graph of $y = \sqrt{x}$.
Solution:



2 Limits

Problem 4. Multiple Choice. Consider the function

$$f(x) = \begin{cases} x^2 & x \text{ rational} \\ -x^2 & x \text{ irrational} \\ \text{undefined} & x = 0 \end{cases}$$

Then

1. There is no a for which $\lim_{x \to a} f(x)$ exists
2. There may be an a for which $\lim_{x \to a} f(x)$ exists, but we can't say what it is without more information
3. $\lim_{x \to a} f(x)$ exists for $a = 0$
4. $\lim_{x \to a} f(x)$ exists for infinitely many a

Solution: Since for any sequence x_n of inputs getting closer to 0, the sequence of outputs $f(x_n)$ runs over either x_n^2 or $-x_n^2$ both of which tend to 0. So the limit $\lim\limits_{x \to 0} f(x)$ exists, and equals 0.

For any other point $a \neq 0$, if we look at a sequence of inputs x_n running over rational numbers, the outputs $f(x_n)$ will be x_n^2 getting closer to a^2. If we look at a sequence of irrational numbers x_n getting closer to a, the sequence of outputs $f(x_n) = -x_n^2$ will get closer to $-a^2$. Since $a \neq 0$, $a^2 \neq -a^2$ and so the limit does not exist.

Therefore, $x = 0$ is the only point where the limit exists.
Problem 5. True or False. The limit $\lim_{x \to a} f(x)$ depends on how $f(a)$ is defined.
Solution: false, the limit only depends on the values near a, not at the point a itself.
Problem 6. True or False. If $f(a)$ is undefined then $\lim_{x \to a}$ cannot exist.
Solution: False, same reason as above.
Problem 7. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ then $\lim_{x \to a} f(x)/g(x)$

1. Does not exist
2. Must exist
3. Not enough information

Solution: 3. is correct. Simple examples show that the limit might exist and might not exist. If we take $f(x) = x^2$ and $g(x) = x$, the limit of $f(x)/g(x)$ exists and equals 0. If we switch the functions, take $f(x) = x$ and $g(x) = x^2$, the limit will not exist.
Problem 8. $\lim_{x \to 0} x^2 \sin(1/x)$

1. Does not exist because no matter how close x gets to 0, there are x's near zero for which $\sin(1/x)$ is 1, and x's for which $\sin(1/x)$ is -1
2. Does not exist because the function value oscillates around 0
3. Does not exist because $1/0$ is undefined
4. Equals 0
5. Equals 1

Solution: Using the Sandwich Theorem, $-x^2 \leq x^2 \sin(1/x) \leq x^2$ and $\lim\limits_{x \to 0} x^2 = \lim\limits_{x \to 0}(-x^2) = 0$.

Problem 9. Find the all the asymptotes of the function $f(x) = \dfrac{1 + x^4}{x^2 - x^4}$.
Solution: Since $\lim\limits_{x \to \infty} \frac{1+x^4}{x^2-x^4} = -1$, there is a horizontal asymptote $y = -1$. Notice that since $\lim\limits_{x \to -\infty} \frac{1+x^4}{x^2-x^4} = -1$ as well, there is only one horizontal asymptote.
Vertical asymptotes correspond to points $x = a$ with $\lim\limits_{x \to a^\pm} f(x) = \pm\infty$. For rational functions this can only happen at the point where the denominators becomes 0. In this case, at the points where $x^2 - x^4 = 0$, i.e. $x = 0, x = \pm 1$. It is clear that $\lim f(x)$ at each of these points is infinite. Thus we have three vertical asymptotes: $x = 0, x = 1, x = -1$.

3 Continuity

Problem 10. You are running a bath but you don't close the tap properly and it is dripping. It drips once per second, each drip raising the level of the bathwater by exactly 1mm.

1. Let f be the function that represents hight of the bathwater at time t. Is $f(x)$ a continuous function?
2. Let g be the function that describes the volume of water as a function of the height of the bathwater. Is $g(x)$ a continuous function?

Solution: 1. Since the height changes in time by jumps (every time the drop falls), the function is not continuous.

2. Function g is continuous, since arbitrary small increments in height give arbitrary small increments in the volume. This function has nothing to do with the water dripping.
Problem 11. You know that

If $f(x)$ is a polynomial function then $f(x)$ is continuous.

Which of the following is true.

1. If $f(x)$ is continuous then $f(x)$ is a polynomial
2. If $f(x)$ is not a polynomial then $f(x)$ is not continuous
3. If $f(x)$ is not continuous then $f(x)$ is not a polynomial
4. All of the above

Solution: If $f(x)$ is not continuous then $f(x)$ is not a polynomial.
Problem 12.

1. Solve the equation $x^2 + 13x + 41 = 1$.
2. Use the IVT to prove that $x^2 + 13x + 41 = \sin x$ has at least 2 solutions between the two roots found above.

Solution: 1. Solving the quadratic equation $x^2 + 13x + 40 = 0$ gives two solutions $x = -8$ and $x = -5$.

2. We know that the value of $x^2 + 13x + 41$ at $x = -8$ and $x = -5$ is 1. Therefore, the values of $f(x) = x^2 + 13x + 41 - \sin x$ at $x = -8$ and $x = -5$ are $1 - \sin(-8)$ and $1 - \sin(-5)$ both positive since $\sin x$ is always ≤ 1.
Now, if we compute the value in the middle between $x = -8$ and $x = -5$, we get $f(-6.5) = -1.25 - \sin(6.5)$ which is negative since $\sin x$ is always between -1 and 1.
Thus, by IVT, there is a point between $x = -8$ and $x = -6.5$ where $f(x) = 0$, and there is a point between $x = -6.5$ and $x = -5$ where $f(x) = 0$.